JHEP HYPER VERSION

Received: *October 15, 1999,* Accepted: *November 8, 1999*

# Instantons, finite $\mathcal{N}=2$ $\mathrm{Sp}(N)$ theories and the AdS/CFT correspondence

**Timothy J. Hollowood**

*Theoretical Division T-8, Los Alamos National Laboratory*
*Los Alamos, NM 87545, USA, and*
*Department of Physics, University of Wales Swansea*
*Swansea, SA2 8PP, UK*
*E-mail:* pyth@skye.lanl.gov

Abstract: We examine ADHM multi-instantons in the conformal $\mathcal{N}=2$ supersymmetric $\mathrm{Sp}(N)$ gauge theory with one anti-symmetric tensor and four fundamental hypermultiplets. We argue that the ADHM construction and measure can also be deduced from purely field theoretic considerations and also from the dynamics of D-instantons in the presence of D3-branes, D7-branes and an orientifold O7-plane. The measure then admits a large-$N$ saddle-point approximation where the D3-branes disappear but the background is changed to $AdS_5 \times S^5/\mathbb{Z}_2$, as expected on the basis of the AdS/CFT correspondence. The large-$N$ measure displays the fractionation of D-instantons at the singularity $S^3 \subset S^5/\mathbb{Z}_2$ and is described for instanton number $k$ by a certain $\mathrm{O}(k)$ matrix model.

Keywords: Solitons Monopoles and Instantons, D-branes, Brane Dynamics in Gauge Theories, Supersymmetry and Duality.

## Contents



## 1. Introduction

In this paper, we shall be concerned with instanton effects in the gauge theory which, in the context of $\mathcal{N} = 4$ supersymmetric SU$(N)$ gauge theory, have a very precise relation to D-instanton effects in the string theory [1, 2, 3, 4] in perfect accord with the AdS/CFT correspondence [5, 6]. This relation can be seen in the dynamics of D-instantons and D3-branes in type-IIB string theory. The world volume theory of $N$ D3-branes is precisely the $\mathcal{N} = 4$ supersymmetric U$(N)$ gauge theory, in the low energy decoupling limit where all energy scales are much smaller than $(\alpha')^{-1/2}$ [3]. Instantons solutions in the gauge theory are identified as D-instantons (or D$(-1)$-branes) bound to the D3-branes [7, 8, 9, 3]. The D-instantons are described by a 'world volume' U$(k)$ matrix model, for $k$ instantons, whose degrees-of-freedom correspond to the ADHM construction of instantons. This theory is the dimensional reduction of six-dimensional $\mathcal{N} = (1, 0)$ supersymmetric gauge theory and has U$(k)$ adjoint and fundamental degrees-of-freedom, the latter arising from strings stretched between the D-instantons and D3-branes. In the decoupling limit this ADHM matrix model is strongly coupled and only certain terms in the action survive. The resulting partition function is identical to the measure on the ADHM moduli space weighted by the instanton action constructed by purely field theoretical methods in [10, 11, 3]. Remarkably, auxiliary scalars introduced to bi-linearize a certain four fermionic collective coordinate interaction are interpreted as the freedom for the D-instantons to be ejected from the D3-branes: they need not be bound!

At large $N$, the ADHM matrix model can be approximated. The idea is to first integrate out the variables corresponding to D$(-1)$-D3 strings, the resulting measure is then amenable to a standard saddle-point analysis. From the point-of-view of the D-instantons the D3-branes disappear to leave a description of D-instantons in a $AdS_5 \times S^5$ background. Instantons are therefore a perfect way to see the AdS/CFT correspondence in action. It may seem surprising that the gauge theory instanton calculation is done at large $N$ with $g^2N \ll 1$ and yet we find perfect agreement with the supergravity regime with $g^2N \gg 1$. Apparently there is a non-renormalization theorem at work, which is natural from the supergravity side of the description [12], but has yet to be proved from the gauge theory side.

Since instantons prove such a powerful tool it is natural to study them in other gauge theories which may or may not have a known dual description. This has been undertaken in the following conformal field theories:

(i) $\mathcal{N}=4$ supersymmetric gauge theory with gauge groups $\mathrm{O}(N)$ and $\mathrm{Sp}(N)$ [13]. In these theories the dual is known to be the type-IIB superstring on $AdS_5 \times \mathbb{RP}^5$ [14, 15].

(ii) $\mathcal{N}=2$ supersymmetric $\mathrm{SU}(N)$ gauge theory with $2N$ fundamental hypermultiplets [16]. In this case the dual string theory is not known.

(iii) 'Orbifold' theories which correspond to certain projections of the $\mathcal{N}=4$ supersymmetric $\mathrm{SU}(N)$ theory [17] by a finite subgroup $\Gamma$ of the $\mathrm{SU}(4)$ $R$-symmetry group of the theory. The resulting theories have either $\mathcal{N}=0,1$ or 2 supersymmetries. In this case the duals are known to be type-IIB superstring theory on $AdS_5 \times S^5/\Gamma$ [18].

There is another class of conformal $\mathcal{N}=2$ theories based on the gauge group $\mathrm{Sp}(N)$.[1] These theories have a $2^{\mathrm{nd}}$-rank anti-symmetric tensor and four fundamental hypermultiplets and are conformal for any $N$. This class of theories is thought to be equivalent to type-IIB theory on $AdS_5 \times S^5/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action fixes an $S^3 \subset S^5$, with four D7-branes wrapped around the $S^3$ and filling $AdS_5$ [19, 20]. What is particularly interesting about his case is that, in common with the $\mathcal{N}=4$ theory, there are detailed predictions for instanton contributions to particular correlation functions [21] which merit investigation from the gauge theory side. This paper sets up the formalism which allows such a comparison. Near the completion of this paper, there appeared a paper by Gava et al. [22] which, as well as setting up the same formalism leading to the large-$N$ instanton measure, performs this comparison and finds perfect agreement with the AdS/CFT correspondence.

Consider the gauge theory in more detail. It is useful to use a mock $\mathcal{N}=4$ notation for the vector multiplet and anti-symmetric tensor hypermultiplet. So the Weyl fermions are labelled $\lambda^A$, $A=1,\ldots,4$, where $A=1,3$ correspond to the vector multiplet, and are therefore adjoint-valued, while $A=2,4$ correspond to the hypermultiplet. All fields can be thought of as $2N \times 2N$ dimensional hermitean matrices subject to

$$X = \pm J X^T J\,, \tag{1.1}$$

[1] For us, $\mathrm{Sp}(N)$ has rank $N$, so $\mathrm{Sp}(1) \equiv \mathrm{SU}(2)$ although it is sometimes denoted $\mathrm{Sp}(2N)$ or even $\mathrm{USp}(2N)$.



JHEP11(1999)012

where $J$ is the usual symplectic matrix

$$J = \begin{pmatrix} 0 & 1_{[N]\times[N]} \\ -1_{[N]\times[N]} & 0 \end{pmatrix}, \tag{1.2}$$

and the upper (lower) sign corresponds to the vector multiplet (anti-symmetric tensor hypermultiplet). With this restriction we see that the adjoint representation of $\mathrm{Sp}(N)$ has dimension $N(2N+1)$ while the anti-symmetric tensor representation has dimension $N(2N-1)$.

In addition, we must add four fundamental hypermultiplets consisting of eight $\mathcal{N}=1$ chiral superfields transforming with respect to an $\mathrm{O}(8)$ flavour symmetry. The theory has vanishing beta function for any $N$ and when $N=1$ the only the fundamental hypermultiplets remain and the theory is equivalent to the conformal $\mathrm{SU}(2)$ theory with four fundamental hypermultiplets.

## 2. The instanton moduli space and measure

The ADHM construction of instanton solutions [23] for gauge group $\mathrm{Sp}(N)$ is described in [24]. However, rather than use the language of quaterions, we find it more in keeping with the string theory connection to think of $\mathrm{Sp}(N)$ as embedded in $\mathrm{SU}(2N)$, and use the ADHM construction of the latter suitably restricted. For a review of the the ADHM construction for unitary groups see [25, 3].

The instanton solution at charge $k$ in $\mathrm{SU}(2N)$, is described by an $(2N+2k)\times 2k$ dimensional matrix $a$, and its conjugate, with the form

$$a = \begin{pmatrix} w_{\dot\alpha} \\ a'_{\alpha\dot\alpha} \end{pmatrix}, \qquad \bar a = \begin{pmatrix} \bar w^{\dot\alpha} & \bar a'^{\dot\alpha\alpha} \end{pmatrix}, \tag{2.1}$$

where $w_{\dot\alpha}$ is a (spacetime) Weyl-spinor-valued $2N\times k$ matrix and $a'_{\alpha\dot\alpha}$ is (spacetime) vector-valued $k\times k$ matrix. The conjugates are defined as

$$\bar w^{\dot\alpha} \equiv (w_{\dot\alpha})^\dagger, \qquad \bar a'^{\dot\alpha\alpha} \equiv (a'_{\alpha\dot\alpha})^\dagger. \tag{2.2}$$

The explicit $\mathrm{SO}(4)$ vector components of $a'$ are defined via $a'_{\alpha\dot\alpha} = a'_n\sigma^n_{\alpha\dot\alpha}$ and the matrices $a'_n$ are restricted to be hermitean: $(a'_n)^\dagger = a'_n$.

Up till now the discussion is valid for gauge group $\mathrm{SU}(2N)$, however, in order to describe gauge group $\mathrm{Sp}(N)$ we only need subject the variables to the additional conditions:

$$\bar w^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}(w_{\dot\beta})^T J, \qquad (a'_{\alpha\dot\alpha})^T = a'_{\alpha\dot\alpha}, \tag{2.3}$$

The remaining $4k(N+k)$ variables are still an over parametrization of the instanton moduli space which is obtained by a hyper-Kähler quotient construction. One first imposes the ADHM constraints:

$$D^{\dot\alpha}{}_{\dot\beta} \equiv \bar w^{\dot\alpha} w_{\dot\beta} + \bar a'^{\dot\alpha\alpha} a'_{\alpha\dot\beta} = \lambda\delta^{\dot\alpha}{}_{\dot\beta}, \tag{2.4}$$

JHEP11(1999)012

where $\lambda$ is an arbitrary constant matrix. The ADHM moduli space is then identified with the space of $a$'s subject to (2.4) modulo the action of an $\mathrm{O}(k)$ symmetry which acts on the instanton indices of the variables as follows

$$w_{\dot\alpha} \to w_{\dot\alpha} U\,, \qquad a'_{\alpha\dot\alpha} \to U a'_{\alpha\dot\alpha} U^T\,, \qquad U \in \mathrm{O}(k)\,. \tag{2.5}$$

Since there are $3k(k-1)/2$ constraints (2.4) and $\mathrm{O}(k)$ has dimension $k(k-1)/2$ the dimension of the ADHM moduli space is $4k(N+1)$.

The final piece of the story is the explicit construction of the self-dual gauge field itself. To this end, we define the matrix

$$\Delta(x) = \begin{pmatrix} w_{\dot\alpha} \\ x_{\alpha\dot\alpha} 1_{[k]\times[k]} + a'_{\alpha\dot\alpha} \end{pmatrix}\,, \tag{2.6}$$

where $x_{\alpha\dot\alpha}$, or equivalently $x_n$, is a point in spacetime. For generic $x$, the $(2N+2k)\times 2N$ dimensional complex-valued matrix $\mathrm{U}(x)$ is a basis for $\ker(\bar\Delta)$:

$$\bar\Delta U = 0 = \bar U \Delta\,, \tag{2.7}$$

where $U$ is orthonormalized according to

$$\bar U U = \ 1_{[2N]\times[2N]}\,. \tag{2.8}$$

The self-dual gauge field is then simply

$$v_n = \bar U \partial_n U\,. \tag{2.9}$$

It is straightforward to show that $v_n$ is anti-hermitean and satisfies $v_n = J(v_n)^T J$ and so is valued in the Lie algebra of $\mathrm{Sp}(N)$.

The fermions in the background of the instanton lead to new Grassmann collective coordinates. Associated to the fermions in the adjoint and anti-symmetric tensor representations, $\lambda^A$, these collective coordinates are described by the $(2N+2k)\times k$ matrices $\mathcal{M}^A$, and their conjugates

$$\mathcal{M}^A = \begin{pmatrix} \mu^A \\ \mathcal{M}'^A_\alpha \end{pmatrix}\,, \qquad \bar{\mathcal{M}}^A = \left(\bar\mu^A\ \bar{\mathcal{M}}'^{\alpha A}\right)\,, \tag{2.10}$$

where $\mu^A$ are $2N\times k$ matrices and $\mathcal{M}'^A_\alpha$ are Weyl-spinor-valued $k\times k$ matrices. The conjugates are defined by

$$\bar\mu^A \equiv (\mu^A)^\dagger\,, \qquad \bar{\mathcal{M}}'^{\alpha A} \equiv (\mathcal{M}'^A_\alpha)^\dagger\,, \tag{2.11}$$

and the constraint $\bar{\mathcal{M}}'^A_\alpha = \mathcal{M}'^A_\alpha$ is imposed.

Up till now the construction of the fermions is valid for the gauge group $\mathrm{SU}(2N)$. For $\mathrm{Sp}(N)$ all we need do is impose the additional conditions

$$\bar\mu^A = (-1)^{A+1}(\mu^A)^T J\,, \qquad (\mathcal{M}'^A_\alpha)^T = (-1)^{A+1}\mathcal{M}'^A_\alpha\,. \tag{2.12}$$

So, for instance, the matrices $\mathcal{M}'^{A}_{\alpha}$ are symmetric, for the vector representation, and anti-symmetric, for the anti-symmetric tensor representation. These symmetry properties can be deduced from the ADHM tensor product construction in [26]. Notice that the different conditions on the collective coordinates breaks the SU(4) symmetry of the mock $\mathcal{N}=4$ notation to $\mathrm{SU}(2)\times\mathrm{U}(1)$, the true $R$-symmetry of the $\mathcal{N}=2$ theory. These fermionic collective coordinates are subject to analogues of the ADHM constraints (2.4):

$$\lambda^A_{\dot\alpha} \equiv \bar w_{\dot\alpha}\mu^A + \bar\mu^A w_{\dot\alpha} + [\mathcal{M}'^{\alpha A}, a'_{\alpha\dot\alpha}] = 0\,. \tag{2.13}$$

One can check that there are consequently $2k(N+1)$ and $2k(N-1)$ fermionic collective coordinates corresponding to $\lambda^A$ for the vector and anti-symmetric tensor representations, respectively. This is in accord with the index theorem that counts the zero eigenvalue solutions to the Dirac equation in the background of the instanton. The quantities

$$\xi^A_\alpha = k^{-1}\,\mathrm{tr}_k\,\mathcal{M}'^A_\alpha\,, \qquad \bar\eta^{\dot\alpha A} = k^{-1}\,\mathrm{tr}_k\,\bar w^{\dot\alpha}\mu^A\,, \tag{2.14}$$

in the vector multiplet, $A=1,3$, are the collective coordinates corresponding to the four supersymmetric and four superconformal zero modes.

Finally we have the fundamental hypermultiplets. They contribute additional fermionic collective coordinates which can be described by a $2N\times 8$ dimensional matrix $\mathcal{K}$. These variables are not subject to any additional ADHM-type constraints. There are no additional bosonic collective coordinates associated to any of the scalar fields in the theory.

At lowest order in $g$, the instanton action, which is the pull-back of the action to the instanton solution, is

$$S^k_{\rm inst} = \frac{8\pi^2 k}{g^2} + S^k_{\rm quad}\,, \tag{2.15}$$

where $S^k_{\rm quad}$ is a certain four-fermion interaction which is induced from the Yukawa interactions of the theory via tree-level scalar exchange. This is where the $\mathcal{N}=4$ labelling comes in handy: the interaction induced by the Yukawa couplings between the vector multiplet and anti-symmetric tensor hypermultiplet is precisely what would arise in the $\mathcal{N}=4$ theory but with the variables restricted as in (2.12). In addition, there is a coupling between a bi-linear in the vector fermionic variables and the hypermultiplet fermionic variables as in [27] arising form the Yukawa couplings between the vector multiplet and fundamental hypermultiplets:

$$S^k_{\rm quad} = \frac{\pi^2}{2g^2}\mathrm{tr}_k\Big[\epsilon_{ABCD}\bar{\mathcal{M}}^A\mathcal{M}^B\mathbf{L}^{-1}\bar{\mathcal{M}}^C\mathcal{M}^D - \mathcal{K}\mathcal{K}^T\mathbf{L}^{-1}(\bar{\mathcal{M}}^1\mathcal{M}^3 - \bar{\mathcal{M}}^3\mathcal{M}^1)\Big]\,. \tag{2.16}$$

Here, $\mathbf{L}$ is the following operator on $k \times k$ matrices:

$$\mathbf{L} \cdot \Omega = \frac{1}{2}\{\Omega, W^0\} + [a'_n, [a'_n, \Omega]] \, , \tag{2.17}$$

where $W^0 \equiv \bar{w}^{\dot{\alpha}} w_{\dot{\alpha}}$. Notice that the fundamental fermionic collective coordinates are only coupled to the adjoint fermionic collective coordinates since there are no Yukawa couplings between the former and the anti-symmetric fermionic collective coordinates.

Notice that the pattern of lifting of fermionic zero modes in the fermion quadrilinear (2.16) leads to a selection rule on the insertions of fermion modes. Suppose the insertions in a correlator involve $n_{\rm adj}$ adjoint, $n_{\rm ast}$ antisymmetric tensor and $n_{\rm f}$ fundamental fermionic collective coordinates, then for a non-trivial instanton contribution we need

$$n_{\rm adj} = n_{\rm ast} + n_{\rm f} \, , \tag{2.18}$$

subject to $n_{\rm adj} \leq 4k(N+1)$, $n_{\rm ast} \leq 4k(N-1)$ and $n_{\rm f} \leq 8k$. In particular, since the action does not lift the 8 exact supersymmetric and superconformal zero-modes we have $n_{\rm adj} \geq 8$.

Later we shall find it essential to bi-linearize this quadrilinear by introducing bosonic variables $\chi_{AB} \equiv -\chi_{BA}$, six $k \times k$ matrices, subject to the reality condition

$$\bar{\chi}^{AB} \equiv (\chi_{AB})^\dagger = \frac{1}{2}\epsilon^{ABCD}\chi_{CD} \, . \tag{2.19}$$

We can write $\chi_{AB}$ as an explicit SO(6) vector $\chi_a$ with components

$$\begin{aligned} &\chi_1 = \sqrt{2}(\chi_{12} + \chi_{34}) \, , \qquad \chi_2 = i\sqrt{2}(\chi_{12} - \chi_{34}) \, , \qquad \chi_3 = \sqrt{2}(\chi_{14} + \chi_{23}) \, , \\ &\chi_4 = i\sqrt{2}(\chi_{14} + \chi_{23}) \, , \qquad \chi_5 = i\sqrt{2}(\chi_{13} - \chi_{24}) \, , \qquad \chi_6 = \sqrt{2}(\chi_{13} + \chi_{24}) \, , \end{aligned} \tag{2.20}$$

normalized so that $\chi_a\chi_a \equiv \epsilon^{ABCD}\chi_{AB}\chi_{CD}$. Taking account of the following symmetry properties

$$(\bar{\mathcal{M}}^{[A}\mathcal{M}^{B]})^T = (-1)^{A+B+1}\bar{\mathcal{M}}^{[A}\mathcal{M}^{B]} \, , \qquad (\mathcal{K}\mathcal{K}^T)^T = -\mathcal{K}\mathcal{K}^T \, , \tag{2.21}$$

the auxiliary variables $\chi_{AB}$ are subject in addition to

$$\chi_{AB} = (-1)^{A+B+1}(\chi_{AB})^T \, . \tag{2.22}$$

So $\chi_a$, $a = 1, \ldots, 4$, are symmetric and $\chi_a$, $a = 5, 6$, are anti-symmetric. These conditions obviously only respect a $\mathrm{SU}(2)\times\mathrm{U}(1)$ subgroup of SU(4), the $R$-symmetry of the $\mathcal{N} = 2$ theory. The transformation we want is then

$$e^{-S^k_{\rm quad}} = (\det{}_s\mathbf{L})^2(\det{}_a\mathbf{L}) \int d\chi \, \exp\big[ -\epsilon^{ABCD}{\rm tr}_k\,\chi_{AB}\mathbf{L}\chi_{CD} + 4\pi i g^{-1}{\rm tr}_k\,\chi_{AB}\Lambda^{AB}\big] \, , \tag{2.23}$$

where $\det_s\mathbf{L}$ and $\det_a\mathbf{L}$ are the determinants of $\mathbf{L}$ evaluated on a basis of symmetric and anti-symmetric matrices, respectively, and we have defined the fermion bi-linear

$$\Lambda^{AB} = \frac{1}{2\sqrt{2}}\bar{\mathcal{M}}^A\mathcal{M}^B - \frac{1}{4\sqrt{2}}\delta^{A2}\delta^{B4}\mathcal{K}\mathcal{K}^T\,. \tag{2.24}$$

The determinant factors in (2.23) are conveniently cancelled by factors that arise from integrating-out pseudo collective coordinates corresponding to the scalars from the vector and anti-symmetric tensor multiplets (see [3]).

The measure on the ADHM moduli space can be deduced from [10, 11, 3] and is simply the 'flat' cartesian measure for all the ADHM variables, including the auxiliary variables $\chi$, with the ADHM constraints (2.4) and (2.13) imposed by explicit delta functions, weighted by the exponential of the action on the right-hand side of (2.23). Up to an overall normalization factor

$$\begin{aligned}\int d\mu^k_{\rm phys}\, e^{-S^k_{\rm inst}} = \frac{1}{{\rm Vol}\ {\rm O}(k)}\int da'\, dw\, d\chi\, d\mathcal{M}'\, d\mu\, d\mathcal{K}\, \times \qquad\qquad (2.25)\\ \times\, \delta(D^{\dot\alpha}_{\ \dot\beta})\,\delta(\lambda^A_{\dot\alpha})\exp\Big[-\epsilon^{ABCD}{\rm tr}_k\chi_{AB}\mathbf{L}\chi_{CD} + 4\pi i g^{-1}{\rm tr}_k\chi_{AB}\Lambda^{AB}\Big]\,,\end{aligned}$$

where the delta functions impose the bosonic and fermionic ADHM constraints (2.4) and (2.13). As in the $\mathcal{N}=4$ case, this measure has the natural interpretation in terms of branes. First of all the $\mathcal{N}=2$ gauge theory can be described by the low energy limit of a set of $N$ D3-branes moving tangent to of one of the four orientifold O7-planes [28, 29] in the type-IIB orientifold of Sen [30]. In this construction there are four D7-branes on top of each of the O7-planes. The gauge theory on the world volume of the D3-branes is precisely our $\mathcal{N}=2$ supersymmetric Sp($N$) gauge theory, with an anti-symmetric tensor hypermultiplet and four fundamental hypermultiplets. The latter fields arise from strings stretched between the D3- and the four D7-branes.

Now on top of this construction we include $k$ D-instantons. The matrix theory on the 'world volume' of the D-instantons will reproduce the ADHM construction and measure described above. In order to see this, it is useful to start from D instantons is flat ten-dimensional space which are described by the dimensional reduction of $\mathcal{N}=1$ supersymmetric gauge theory in ten dimensions. The Lorentz group in ten-dimensional euclidean space is broken to $H = {\rm SO}(4)_1\times{\rm SO}(4)_2\times{\rm SO}(2)\subset{\rm SO}(10)$, where the first factor corresponds to the directions along the D3-brane world-volume, the second factor to the directions in the D7/O7 world-volume orthogonal to the D3-branes and finally the last factor corresponds to the two directions orthogonal to the D7/O7 world volume. The components of ten-dimensional gauge field decompose as

$$\mathbf{10}\to(\mathbf{4},\mathbf{1},\mathbf{1})^s\oplus(\mathbf{1},\mathbf{4},\mathbf{1})^s\oplus(\mathbf{1},\mathbf{1},\mathbf{2})^a \tag{2.26}$$

corresponding in the ADHM construction to $a'_n$, $\chi_a$, $a=1,\ldots,4$ and $\chi_a$, $a=5,6$, respectively. The superscripts in (2.26) label the projections on the U($2N$) valued variables imposed in the orientifold background, where $s$ means symmetric and $a$ means anti-symmetric.

In order to describe the fermions we consider the covering group of $H$, $\bar{H} = \mathrm{SU}(2)_{L_1} \times \mathrm{SU}(2)_{R_1} \times \mathrm{SU}(2)_{L_2} \times \mathrm{SU}(2)_{R_2} \times \mathrm{U}(1)$. The Majorana-Weyl fermion of the ten-dimensional theory decomposes as

$$\mathbf{16} \to (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})_1^s \oplus (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2})_1^a \oplus (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})_{-1}^s \oplus (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})_{-1}^a \tag{2.27}$$

under $\bar{H}$. Again the superscripts indicate the projection imposed by the orientifold background. The first two components are identified with the ADHM variables $\mathcal{M}'^A_\alpha$, for $A = 1,3$ and $2,4$, respectively, and the latter two with some additional variables $\lambda^{\dot\alpha}_A$ whose rôle will emerge shortly. Notice that the subgroup $\mathrm{SU}(2)_{L_2} \times \mathrm{U}(1) \subset \bar{H}$ is identified with the $R$-symmetry of the original $\mathcal{N} = 2$ gauge theory.

The remaining ADHM variables correspond to the presence of the D3- and D7-branes. For instance $w_{\dot\alpha}$ and $\mu^A$ are associated to strings stretched between the D-instantons and the D3-branes and finally the variables $\mathcal{K}$ are associated to strings stretched between the D-instantons and D7-branes (there are no bosonic variables associated to these strings). We shall not write down the full action for this theory, but it is a simple generalization to include $\mathcal{K}$ of that for the $\mathcal{N} = 4$ theory written down in [3]. However, as explained in [3] it is useful to introduce additional bosonic auxiliary $k \times k$ matrices $D^{\dot\alpha}{}_{\dot\beta}$ transforming in a triplet of $\mathrm{SU}(2)_{R_1}$, and in the present context anti-symmetric, whose significance will emerge shortly. Now in the decoupling limit $\alpha' \to 0$, the coupling constant of the D-instanton 'world volume' matrix theory goes to infinity which effectively removes certain terms from the action. It is straightforward to show that the remaining action is precisely that on the right-hand side of (2.26) and the variables $D^{\dot\alpha}{}_{\dot\beta}$ and $\lambda^{\dot\alpha}_A$ act as Lagrange multipliers that impose the bosonic and fermionic ADHM constraints (2.4) and (2.13), respectively.

## 3. The large-$N$ measure

We now want to take find an approximation of the ADHM measure valid in the large $N$-limit. The procedure is by now well documented [3, 16, 17].

First of all it is advantageous to write the measure in terms of a set of gauge invariant variables defined in terms of a $2k \times 2k$ dimensional matrix

$$W^{\dot\alpha}{}_{\dot\beta} = \bar{w}^{\dot\alpha} w_{\dot\beta} \,. \tag{3.1}$$

From this we define the following four $k \times k$ matrices

$$W^0 = \bar{w}^{\dot\alpha} w_{\dot\alpha} \,, \qquad W^c = (\tau^c)^{\dot\beta}{}_{\dot\alpha} \bar{w}^{\dot\alpha} w_{\dot\beta} \,. \tag{3.2}$$

The matrix $W^0$ is symmetric while the three matrices $W^c$ are anti-symmetric. We now change variables from $w_{\dot\alpha}$ to the gauge invariant variables above and the gauge transformations acting on the solution which parameterize a coset. Since the measure is used to integrate gauge invariant quantities we integrate over the gauge coset which yields a volume factor. Up to numerical factors[2] and for $N \geq k$ we have

$$\int_{\text{coset}} dw \sim \int (\det_{2k} W)^{N-k^2/2+1/4}\, dW^0 \prod_{c=1,2,3} dW^c\,. \tag{3.3}$$

In terms of the gauge invariant coordinates the ADHM constraints (2.4) are linear:

$$W^c = i\bar{\eta}^c_{nm}[a'_n, a'_m]\,, \tag{3.4}$$

where $\bar{\eta}^c_{nm}$ is a 't Hooft eta-symbol defined in [3], and hence can be trivially integrated out.

On the fermionic side we need to integrate out the superpartners of the gauge degrees-of-freedom. To isolate these variables we expand

$$\mu^A = w_{\dot{\alpha}}\zeta^{\dot{\alpha}A} + w_{\dot{\alpha}}\sigma^{\dot{\alpha}A} + \nu^A\,, \tag{3.5}$$

where the $\nu^A$ modes — the ones we need to integrate out — form a basis for the kernel of $\bar{w}^{\dot{\alpha}}$, i.e. $\bar{w}^{\dot{\alpha}}\nu^A = 0$, The remaining variables are chosen to have the symmetry properties

$$(\zeta^{\dot{\alpha}A})^T = (-1)^{A+1}\zeta^{\dot{\alpha}A}\,, \qquad (\sigma^{\dot{\alpha}A})^T = (-1)^A\sigma^{\dot{\alpha}A}\,. \tag{3.6}$$

Notice that the $\nu^A$ variables decouple from the fermionic ADHM constraints (2.13). The change of variables from $\mu^A$ to $\{\zeta^A, \sigma^A, \nu^A\}$ involves a jacobian:

$$\int \prod_{A=1,2,3,4} d\mu^A = \int (\det_{2k} W)^{-2k} \prod_{A=1,2,3,4} d\zeta^A\, d\sigma^A\, d\nu^A\,. \tag{3.7}$$

We can now integrate out the $\nu^A$ variables:

$$\int \prod_{A=1,2,3,4} d\nu^A \exp\left[\sqrt{8}\pi i g^{-1}\mathrm{tr}_k\, \chi_{AB}\bar{\nu}^A\nu^B\right] = 2^{6k(N-k)}(\pi/g)^{4k(N-k)}\left(\det_{4k}\chi\right)^{N-k}\,. \tag{3.8}$$

Furthermore, the fermionic ADHM constraints (2.13) can then be used to integrate-out the variables $\sigma^{\dot{\alpha}A}$. The resulting expression is rather cumbersome and is, in any case, not needed at this stage. Fortunately, in the large $N$ limit it simplifies considerably.

We now proceed to take a large-$N$ limit of the measure using a steepest descent method. As explained in [3] it is useful to scale $\chi_{AB} \to \sqrt{N}\chi_{AB}$ because this exposes the three terms which contribute to the large-$N$ 'saddle-point' action:

$$S_{\text{eff}} = -\log\det_{2k} W - \log\det_{4k}\chi + \mathrm{tr}_K\, \chi_a \mathbf{L}\chi_a\,. \tag{3.9}$$

The first and second terms come from (3.3) and (3.8) while the final term comes from (2.26).

[2]For later use, the numerical pre-factor goes like $N^{-2kN+k^2-k/2}$ at large $N$.

We can now perform a steepest descent approximation of the measure which involves finding the minima of the effective action with respect to the variables $\chi_{AB}$, $W^0$ and $a'_n$. The resulting saddle-point equations are[3]

$$\begin{aligned}
\epsilon^{ABCD}\left(\mathbf{L}\cdot\chi_{AB}\right)\chi_{CE} &= \frac{1}{2}\delta^D_E\,1_{[K]\times[K]}\,,\\
\chi_a\chi_a &= \frac{1}{2}(W^{-1})^0\,,\\
[\chi_a,[\chi_a,a'_n]] &= i\bar\eta^c_{nm}[a'_m,(W^{-1})^c]\,.
\end{aligned}\tag{3.10}$$

where we have introduced the matrices

$$(W^{-1})^0=\mathrm{tr}_2\,W^{-1}\,,\qquad (W^{-1})^c=\mathrm{tr}_2\,\tau^cW^{-1}\,.\tag{3.11}$$

We note that the expression for the effective action (3.9) and the saddle point equations (3.10) look identical to those derived in [3] for the $\mathcal{N}=4$ theory, however, the difference is the symmetry properties of the variables $\{a'_n,W^0,\chi_a\}$ which will have to be taken into account. As in the $\mathcal{N}=4$ case we look for a solution with $W^c=0$, $c=1,2,3$, which means that the instantons are embedded in mutually commuting SU(2) subgroups of the gauge group. In this case equations (3.10) are equivalent to

$$[a'_n,a'_m]=[a'_n,\chi_a]=[\chi_a,\chi_b]=0\,,\qquad W^0=\frac{1}{2}(\chi_a\chi_a)^{-1}\,.\tag{3.12}$$

The final equation can be viewed as giving the value of $W^0$, whose eigenvalues are the instanton sizes squared at the saddle-point. Clearly $\chi_a\chi_a$ and $W^0$ must be non-degenerate.

The solution space to these equations (up to the auxiliary O($k$) symmetry) is composed of two distinct branches. On the first branch

$$\begin{aligned}
a'_n &= \mathrm{diag}\big(-X^1_n,\ldots,-X^k_n\big)\,,\\
\chi_a &= \begin{cases}\mathrm{diag}\big(\rho_1^{-1}\hat\Theta^1_a,\ldots,\rho_k^{-1}\hat\Theta^k_a\big) & a=1,\ldots,4\,,\\ 0 & a=5,6\,,\end{cases}
\end{aligned}\tag{3.13}$$

where $\hat\Theta_a$ is a unit four-vector, i.e. $\hat\Theta_a\equiv 0$ for $a=5,6$. A second branch of solutions only exists for even instanton number $2k$ and is, in block form,

$$a'_n=\mathrm{diag}\big(-X^1_n,\ldots,-X^k_n\big)\otimes 1_{[2]\times[2]}\,,\tag{3.14}$$

$$\chi_a=\mathrm{diag}\big(\rho_1^{-1}\hat\Omega^1_a,\ldots,\rho_k^{-1}\hat\Omega^k_a\big)\otimes\begin{cases}1_{[2]\times[2]} & a=1,\ldots,4\,,\\ \tau^2 & a=5,6\,,\end{cases}\tag{3.15}$$

[3] As usual we shall frequently swap between the two representations $\chi_a$ and $\chi_{AB}$ for SO(6) vectors.



JHEP11(1999)012

where $\hat{\Omega}_a$ is a unit six-vector. Notice that the sign of $\hat{\Omega}^i_a$, $a = 5,6$, can be reversed by an $\mathrm{O}(2k)$ transformation $\tau^1$. So physically $\hat{\Omega}^i_a$ is coordinate on the orbifold $S^5/\mathbb{Z}_2$. In these solutions $X^i_n$ are the position of the individual instantons in $\mathbb{R}^4$ and $\rho_i$ are their scale sizes.

The first branch can be interpreted as parameterizing, via $\{X^i_n, \rho_i, \hat{\Theta}^i_a\}$, the position of $k$ objects, the D-instantons, moving in $AdS_5 \times S^3$, where the $S^3$ is precisely the orbifold singularity of $S^5/\mathbb{Z}_2$ of the dual type-IIB superstring background. Since these instantons are confined to the singularity they are fractional [31, 32, 33]. The second branch describes how the fractional D-instantons confined to the singularity on the first branch can move off in pairs to explore the whole of $S^5/\mathbb{Z}_2$ parametrized by $\hat{\Omega}^i_a$. Of course there are also mixed branches consisting of fractional instantons on, and 'normal' instanton off, the singularity. However, we shall not need these more general solutions in our analysis.

In principle, in order to do a saddle-point analysis we have to expand the effective action around these general solutions to sufficient order to ensure that the fluctuation integrals converge. In general because the gaussian form has zeros whenever two D-instantons coincide one has to go to quartic order in the fluctuations. Fortunately, as explained in [3], we do not need to expand about these most general solution to the saddle-point equations to quartic order since this is equivalent to expanding to the same order around the most degenerate solution where all the D-instantons are at the same point. The resulting quartic action has flat directions corresponding to the relative positions of the D-instantons. However, when the fermionic integrals are taken into account the integrals over these relative positions turn out to be convergent and hence these degrees-of-freedom should be viewed as fluctuations around rather than facets of the maximally degenerate solution. The variables left un-integrated, since they are not convergent, are the centre-of-mass coordinates. In the present situation, as in the orbifold theories [17], there is an important further implication: the number of centre-of-mass coordinates (corresponding to the non-damped integrals) can depend on exactly what fermionic insertions are made into measure since this affects the convergence properties of the bosonic integrals. We will see that this is closely connected with the different branches of the solution space.

For the first branch of solutions, the maximally degenerate solution is (3.13) with all the instantons at the same point:

$$\begin{aligned} a'_n &= -X_n 1_{[k]\times[k]}\,, \\ \chi_a &= \begin{cases} \rho^{-1}\hat{\Theta}_a 1_{[k]\times[k]} & a = 1,\ldots,4\,, \\ 0 & a = 5,6\,. \end{cases} \end{aligned} \tag{3.16}$$

For the second branch with instanton number $2k$, the maximally degenerate solution is (3.15) with all the instantons at the same point:

$$\begin{aligned} a'_n &= -X_n 1_{[2k]\times[2k]}\,, \\ \chi_a &= \rho^{-1}\hat{\Omega}_a 1_{[k]\times[k]} \otimes \begin{cases} 1_{[2]\times[2]} & a = 1,\ldots,4\,, \\ \tau^2 & a = 5,6\,. \end{cases} \end{aligned} \tag{3.17}$$

In (3.16) $\hat{\Theta}_a$ parametrizes $S^3$, whereas in (3.17), $\hat{\Omega}_a$ parametrizes $S^5/\mathbb{Z}_2$.

The expansion of the effective action to quartic order around these solution can be deduced from the analysis of [3], although the first case describing the fractional D-instantons is somewhat simpler and we describe it first. After integrating-out the fluctuations in the variables $W^0$ which are lifted at gaussian order, as in [3], the quartic fluctuations are governed by the action which looks identical to the action of ten-dimensional $\mathcal{N}=1$ supersymmetric $\mathrm{U}(k)$ gauge theory dimensionally reduced to zero dimensions:

$$S_{\mathrm{b}} = -\frac{1}{2}\mathrm{tr}_k\,\big[A_\mu, A_\nu\big]^2\,, \tag{3.18}$$

where the gauge field has components

$$A_\mu = N^{1/4}\big(\rho^{-1}a'_n, \rho\chi_a\big)\,. \tag{3.19}$$

However, the components of the gauge field are subject to

$$\begin{aligned} (A_\mu)^T &= A_\mu\,, && \text{for } \mu = 1,\ldots,8\,, \\ (A_\mu)^T &= -A_\mu\,, && \text{for } \mu = 9,10\,. \end{aligned} \tag{3.20}$$

In other words, the matrix model only has $\mathrm{O}(k)\subset\mathrm{U}(k)$ symmetry. The centre-of-mass parameters of the maximally degenerate solution, $X_n$ and $\rho^{-1}\hat{\Theta}_a$, correspond to the trace parts of $A_\mu$, $\mu=1,\ldots,8$, and these decouple from the action.

Now we turn to the fermionic sector which are coupled to the bosonic variables in (2.23). First of all we fulfill our promise to deal with the fermionic ADHM constraints. To leading order in $1/N$, these constraints read

$$2\rho^2\sigma^A_{\dot{\alpha}} = -\frac{1}{2}[\delta W^0, \zeta^A_{\dot{\alpha}}] - [\mathcal{M}'^{\alpha A}, a'_{\alpha\dot{\alpha}}]\,. \tag{3.21}$$

So the integrals over the $\sigma^{\dot{\alpha}A}$ variables soak up the delta-functions imposing the fermionic ADHM constraints, as promised. In (3.21), $\delta W^0$ are the fluctuations in $W^0$, all of which are lifted at gaussian order. Due to a cross term we can effectively replace $\delta W^0$ with $-4\rho^3\hat{\Theta}\cdot\chi$ at leading order (see [3]). Collecting all the leading order terms, the fermion couplings are

$$\begin{aligned} S_{\mathrm{f}} = i\Big(\frac{8\pi^2 N}{g^2}\Big)^{1/2}\mathrm{tr}_k\Big[&-2\rho^2(\hat{\Theta}\cdot\chi)\hat{\Theta}_{AB}\zeta^{\dot{\alpha}A}\zeta^B_{\dot{\alpha}} + \rho^{-1}\hat{\Theta}_{AB}\big[a'_{\alpha\dot{\alpha}}, \mathcal{M}'^{\alpha A}\big]\zeta^{\dot{\alpha}B} + \\ &+ \chi_{AB}\big(\rho^2\zeta^{\dot{\alpha}A}\zeta^B_{\dot{\alpha}} + \mathcal{M}'^{\alpha A}\mathcal{M}'^B_{\alpha}\big) - \frac{1}{2}\chi_{24}\mathcal{K}\mathcal{K}^T\Big]\,. \end{aligned} \tag{3.22}$$

The fermionic variables $\mathcal{M}'^{A}_{\alpha}$ and $\zeta^{\dot\alpha A}$ can be amassed into two SO(8) spinors $\Psi$ and $\Phi$:

$$\begin{aligned}\Psi &= \sqrt{\frac{\pi}{2g}}N^{1/8}\left(\rho^{-1/2}\mathcal{M}'^{1}_{\alpha},\rho^{-1/2}\mathcal{M}'^{3}_{\alpha},\rho^{1/2}\zeta^{\dot\alpha 1},\rho^{1/2}\zeta^{\dot\alpha 3}\right),\\ \Phi &= \sqrt{\frac{\pi}{2g}}N^{1/8}\left(\rho^{-1/2}\mathcal{M}'^{2}_{\alpha},\rho^{-1/2}\mathcal{M}'^{4}_{\alpha},\rho^{1/2}\zeta^{\dot\alpha 2},\rho^{1/2}\zeta^{\dot\alpha 4}\right),\end{aligned} \tag{3.23}$$

transforming as an $\mathbf{8}_s$ and $\mathbf{8}_{\bar s}$, respectively, and as $k\times k$ matrices subject to

$$\Psi^T=\Psi\,,\qquad \Phi^T=-\Phi\,. \tag{3.24}$$

We also define rescaled fundamental collective coordinates

$$\mathcal{K}\to\sqrt{\frac{4g}{\pi}}N^{-1/8}\mathcal{K}\,. \tag{3.25}$$

In terms of the new variables, the fermion couplings can be written in an elegant way as

$$S_{\rm f}=i\,{\rm tr}_k\Big(\Psi\Gamma_{\hat\mu}[A_{\hat\mu},\Phi]+\Psi[A_9-iA_{10},\Psi]+\Phi[A_9+iA_{10},\Phi]-(A_9+iA_{10})\mathcal{K}\mathcal{K}^T\Big)\,, \tag{3.26}$$

with the appropriate SO(8) inner products between the spinors. Here $\Gamma_{\hat\mu}$, $\hat\mu=1,\dots,8$ is a representation of the $D=8$ Clifford algebra which depends upon $\hat\Theta$.[4]

The couplings (3.26) do not involve the trace parts of $\Psi$ corresponding to the eight supersymmetric and superconformal zero-modes of the instanton, which we denoted previously as $\xi^A_{\dot\alpha}$ and $\bar\eta^{\dot\alpha A}$, for $A=1,3$. Separating out the integrals over these variables and the bosonic centre-of-mass variables, the measure at large $N$ has the form[5]

$$\int d\mu^k e^{-S^k_{\rm inst}}\Big|_{1^{\rm st}\ {\rm branch}}\underset{N\to\infty}{=}g^4Ne^{2\pi ik\tau}\int\frac{d^4X\,d\rho}{\rho^5}\,d^3\hat\Theta\prod_{A=1,3}d^2\xi^A\,d^2\bar\eta^A\cdot\mathcal{Z}_{{\rm O}(k)}\,, \tag{3.27}$$

where $\mathcal{Z}_{{\rm O}(k)}$ is the partition function of an O$(k)$ matrix model:

$$\mathcal{Z}_{{\rm O}(k)}=\frac{1}{{\rm Vol\ O}(k)}\int d\hat A\,d\hat\Psi\,d\Phi\,d\mathcal{K}\,e^{-S(A_\mu,\Psi,\Phi,\mathcal{K})}\,, \tag{3.28}$$

where the hat indicates the traceless parts and $S(A_\mu,\Psi,\Phi,\mathcal{K})=S_{\rm b}+S_{\rm f}$. This matrix model is identical to the one written down in [21] and describes the dynamics

[4] The construction of this Clifford algebra can be deduced from the similar construction in $D=10$ for the $\mathcal{N}=4$ theory in [3].

[5] We have kept track of power of $N$ and $g$ but not other numerical factors.



JHEP11(1999)012

of $k$ D-instantons in flat space. In other words, at large $N$ the D3-branes have disappeared explicitly in the description of D-instanton dynamics and their only effect is to change the centre-of-mass measure from $\mathbb{R}^{10}$ to $AdS_5 \times S^3$, along with the 8 supersymmetric and superconformal fermionic integrals which are associated to the 8 symmetries broken by the instanton. One can show that the integrals over all the bosonic variables in the definition of $\mathcal{Z}_{\mathrm{O}(k)}$ are actually convergent so that $\mathcal{Z}_{\mathrm{O}(k)}$ is some finite numerical factor (this is important for getting the $N$ counting of the answer correct).

Now we turn to the expansion around the maximally degenerate solution for the second branch of solutions (3.15) with instanton number $2k$. In this case the solution is not proportional to the identity and many of the fluctuations, those which do not commute $\chi_a$, for $a = 5, 6$, are lifted at gaussian order. In fact it is not difficult to see that the resulting leading order term can be deduced from the $\mathrm{O}(2k)$ matrix model constructed above by expanding around $\chi_a = \rho^{-1}\hat{\Omega}_a 1_{[k]\times[k]}\otimes\tau^2$, $a = 5, 6$. The solution only commutes with a subgroup $\mathrm{U}(k) \subset \mathrm{O}(2k)$, and so the resulting model only has $\mathrm{U}(k)$ symmetry. After a suitable gauge fixing (described in the context of the orbifold models in [17]) and after integrating-out the gaussian fluctuations, which include all the fundamental collective coordinates $\mathcal{K}$, one is left with the partition function of a $\mathrm{U}(k)$ matrix model which is identical to that which appears in the $\mathcal{N} = 4$ calculation namely ten-dimensional $\mathcal{N} = 1$ gauge theory dimensionally reduced to zero dimensions. The $\mathrm{U}(k)$ gauge field $A'_\mu$ is embedded in the $\mathrm{O}(2k)$ model variables $A_\mu$ as

$$A_\mu = A'_\mu \otimes \begin{cases} 1_{[2]\times[2]} & \mu = 1, \dots, 8\,, \\ \tau^2 & \mu = 9, 10\,. \end{cases} \tag{3.29}$$

with a similar relation for the $\mathrm{U}(k)$ fermions:

$$\Psi = \Psi' \otimes 1_{[2]\times[2]}\,, \qquad \Phi = \Phi' \otimes \tau^2\,. \tag{3.30}$$

The 16 fermions $\Psi'$ and $\Phi'$ combine into the 16 component Majorana Weyl fermion of the ten-dimensional theory. The trace parts of the gauge field $A'_\mu$ separate out to give an integral over $AdS_5 \times S^5/\mathbb{Z}_2$ along with 16 fermionic integrals which include the 8 supersymmetric and superconformal zero modes along with the 8 components of $\Phi$ proportional to $1_{[k]\times[k]} \otimes \tau^2$, which for conformity of notation we denote $\xi^A_\alpha$ and $\bar{\eta}^{\dot{\alpha}A}$, for $A = 2, 4$. The final expression is

$$\int d\mu^{2k}_{\mathrm{phys}} e^{-S^{2k}_{\mathrm{inst}}}\Big|_{2^{\mathrm{nd}}\ \mathrm{branch}} \underset{N\to\infty}{=} g^8\sqrt{N}e^{4\pi ik\tau}\int \frac{d^4X\, d\rho}{\rho^5}\, d^5\hat{\Omega} \prod_{A=1,\dots,4} d^2\xi^A\, d^2\bar{\eta}^A \cdot \mathcal{Z}_{\mathrm{SU}(k)}\,, \tag{3.31}$$

which is identical to the measure in the $\mathcal{N} = 4$ case. Again, it is important that bosonic integrals in the definition of the partition function $\mathcal{Z}_{\mathrm{SU}(k)}$ are all convergent and this factor is some overall numerical factor. Our analysis makes it clear that the second branch is actually contained within the more general first branch. The key point is that the convergence of the integrals of the bosonic variables in the $\mathrm{O}(2k)$ matrix model depends on the 'fermionic context', i.e. on what fermion insertions are made into the partition function. The second branch corresponds to inserting the 8 modes $\xi^A_\alpha$ and $\bar{\eta}^{\dot{\alpha}A}$, $A = 2, 4$, into the $\mathrm{O}(2k)$ partition function. The integrals over the components of $A_\mu$, for $\mu = 9, 10$, proportional to $1_{[k]\times[k]} \otimes \tau^2$, are not damped after taking into account the now smaller number of fermionic integrals. These divergent integrals need to be separated out as additional centre-of-mass coordinates leading to an integral over $S^5/\mathbb{Z}_2$ rather than $S^3$.

**Note added:** near the completion of this work there appeared a paper by Gava et al. [22]. That paper has identical conclusions to the present work but goes much further and provides a very detailed comparison between instanton contributions to certain correlators in the gauge theory and the string theory. The results are in perfect agreement with the AdS/CFT correspondence.



## Acknowledgments

I would like to thank Michael Gutperle for very helpful conversations.

## References

[1] M. Bianchi, M.B. Green, S. Kovacs and G. Rossi, *Instantons in supersymmetric Yang-Mills and d instantons in IIB superstring theory*, *J. High Energy Phys.* **08** (1998) 013 [hep-th/9807033].

[2] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multiinstantons and Maldacena's conjecture*, *J. High Energy Phys.* **06** (1999) 023 [hep-th/9810243].

[3] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-instanton calculus and the AdS/CFT correspondence in $N = 4$ superconformal field theory*, *Nucl. Phys.* **B 552** (1999) 88 [hep-th/9901128].

[4] M.B. Green, *Interconnections between type-II superstrings, M-theory and $N = 4$ supersymmetric Yang-Mills*, hep-th/9903124.

[5] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[6] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, hep-th/9905111.

[7] E. Witten, *Small instantons in string theory*, *Nucl. Phys.* **B 460** (1996) 541 [hep-th/9511030].

[8] M.R. Douglas, *Branes within branes*, hep-th/9512077.

[9] M.R. Douglas, *Gauge fields and D-branes*, *J. Geom. Phys.* **28** (1998) 255 [hep-th/9604198].

[10] N. Dorey, V.V. Khoze and M.P. Mattis, *Supersymmetry and the multi - instanton measure*, *Nucl. Phys.* **B 513** (1998) 681 [hep-th/9708036].

[11] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Supersymmetry and the multi - instanton measure II: from $N = 4$ to $N = 0$*, *Nucl. Phys.* **B 519** (1998) 470 [hep-th/9709072].

[12] R. Gopakumar and M.B. Green, *Instantons and nonrenormalization in AdS/CFT*, hep-th/9908020.

[13] T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Instantons in $N = 4$ $\mathrm{Sp}(N)$ and $\mathrm{SO}(N)$ theories and the AdS/CFT correspondence*, hep-th/9910118.

[14] Z. Kakushadze, *Gauge theories from orientifolds and large-N limit*, *Nucl. Phys.* **B 529** (1998) 157 [hep-th/9803214].

[15] Z. Kakushadze, *On large-N gauge theories from orientifolds*, *Phys. Rev.* **D 58** (1998) 106003 [hep-th/9804184].

[16] T.J. Hollowood, V.V. Khoze and M.P. Mattis, *Summing the instanton series in $N = 2$ superconformal large-N QCD*, *J. High Energy Phys.* **10** (1999) 019 [hep-th/9905209].

[17] T.J. Hollowood and V.V. Khoze, *ADHM and D instantons in orbifold AdS/CFT duality*, hep-th/9908035.

[18] S. Kachru and E. Silverstein, *4-D conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183].

[19] A. Fayyazuddin and M. Spalinski, *Large-N superconformal gauge theories and supergravity orientifolds*, *Nucl. Phys.* **B 535** (1998) 219 [hep-th/9805096].

[20] O. Aharony, A. Fayyazuddin and J. Maldacena, *The large-N limit of $N = 2$, $N = 1$ field theories from three- branes in F-theory*, *J. High Energy Phys.* **07** (1998) 013 [hep-th/9806159].

[21] M. Gutperle, *Heterotic/type-I duality, D instantons and a $N = 2$ AdS/CFT correspondence*, hep-th/9905173.

[22] E. Gava, K.S. Narain and M.H. Sarmadi, *Instantons in $N = 2$ $\mathrm{Sp}(N)$ superconformal gauge theories and the AdS/CFT correspondence*, hep-th/9908125.

[23] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of instantons*, *Phys. Lett.* **A 65** (1978) 185.

[24] E.F. Corrigan, D.B. Fairlie, S. Templeton and P. Goddard, *A Green's function for the general selfdual gauge field*, *Nucl. Phys.* **B 140** (1978) 31.

[25] V.V. Khoze, M.P. Mattis and M.J. Slater, *The instanton hunter's guide to supersymmetric* SU($N$) *gauge theories*, *Nucl. Phys.* **B 536** (1998) 69 [hep-th/9804009].

[26] E. Corrigan, P. Goddard and S. Templeton, *Instanton Green's functions and tensor products, Nucl. Phys.* **B 151** (1979) 93.

[27] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi instanton calculus in $N = 2$ supersymmetric gauge theory II: coupling to matter*, *Phys. Rev.* **D 54** (1996) 7832 [hep-th/9607202].

[28] T. Banks, M.R. Douglas and N. Seiberg, *Probing F-theory with branes*, *Phys. Lett.* **B 387** (1996) 278 [hep-th/9605199].

[29] M.R. Douglas, D.A. Lowe and J.H. Schwarz, *Probing F-theory with multiple branes*, *Phys. Lett.* **B 394** (1997) 297 [hep-th/9612062].

[30] A. Sen, *F-theory and orientifolds*, *Nucl. Phys.* **B 475** (1996) 562 [hep-th/9605150].

[31] M. R. Douglas and G. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[32] M.R. Douglas, *Enhanced gauge symmetry in M(atrix) theory*, *J. High Energy Phys.* **07** (1997) 004 [hep-th/9612126].

[33] D.-E. Diaconescu, M.R. Douglas and J. Gomis, *Fractional branes and wrapped branes*, *J. High Energy Phys.* **02** (1998) 013 [hep-th/9712230].